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Moduli Fields in Warped Compactifications

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We discuss the effective action of moduli fields in warped brane-world compactifications. For definiteness, a two-brane model with a bulk dilaton field and a power-law warp factor is considered. After deriving the classical four-dimensional effective action for the moduli, we present the calculation of the one-loop effective potential induced by bulk fields. A detailed discussion of renormalization is given, with emphasis on the local worldsheet operators which are generated. Finally, we outline the possible role of these operators in the stabilization of the moduli.

KEY WORDS: moduli fields; warped compactifications.

1. INTRODUCTION

Brane-world scenarios, where two or more parallel branes of dimension 3+1 are embedded in a "bulk" of larger dimension, have recently been used in order to construct models of considerable phenomenological interest, both in particle physics and in cosmology (Antoniadis *et al.*, 1998; Arakani-Hamed *et al.*, 1988, 1999; Chiba, 2000; Garriga and Tanaka, 1999; Khoury et al., 2001; Lukas *et al.*, 1999; Randall and Sundrum, 1999; Shiromizu *et al.*, 2000). Usually, these models admit an effective four-dimensional (4D) description at low energies, where the distances between branes are represented by scalar fields $\varphi_i(x^\mu)(\mu = 0, \ldots, 3)$ called moduli.

An important example is the Randall–Sundrum I scenario (RS) (Randall and Sundrum, 1999), where the bulk is a slice of a five-dimensional (5D) anti-De Sitter space (AdS) bounded by two branes of opposite tension. Matter fields may be restricted to live on the branes, or, as in many extensions of the original model, some of them are allowed to live in the bulk too (Gherghetta and Pomarol, 2000; Goldberger and Wise, 1999a). In RS there is a single modulus, called the *radion*, related to the thickness of the AdS slice. It is given by $\psi(x^\mu) \sim m_{\rm p} \exp[-d(x^\mu)/\ell]$ where d is the physical interbrane distance, ℓ is the AdS, curvature radius, and $m_{\rm p}$

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is the Planck mass. In the 5D theory all fields are assumed to have physical masses comparable to m_p, but the 4D effective mass $m \sim m_p \langle \psi \rangle$ of fields living in (or near) the negative tension brane, is much smaller. This is due to a *redshift* effect induced by the AdS geometry between the branes. The exponential dependence of the masses m on $\langle d \rangle$ can easily explain a large hierarchy between the Planck scale and the electroweak scale, without the need of invoking very large numbers. This is of course one of the main assets of the RS construction.

Usually, moduli fields are introduced by substituting certain integration constants in the classical solutions (e.g. the interbrane distances) by slowly varying fields. If the solutions exist for a continuous range of the integration constants, the corresponding moduli are massless. This corresponds, for instance, to the situation where there is no interaction between branes. Phenomenologically, however, massless fields are undesirable because they give rise to long range scalar forces which are severely constrained by observations. Another point to consider is that through their couplings to matter, the moduli tend to evolve cosmologically, causing time variations of the fundamental constants (see, e.g., Brax *et al.*, 2002, for a recent discussion of this issue).

Therefore it is of some interest to investigate possible stabilization mechanisms for the moduli. A stabilization can be achieved by introducing new interactions determining the equilibrium distances between the branes. In the RS scenario, a very elegant mechanism has been advocated by Goldberger and Wise (GW) (Goldberger and Wise, 1999b; Tanaka and Montes, 2000). In the GW mechanism, an additional bulk scalar field with suitable bulk and brane potentials holds the branes in place, and gives the radion a mass slightly below the TeV scale, making it potentially accessible to collider experiments (see, e.g., Kribs, 2001, for a recent review).

An alternative possibility is that even if the moduli are classically massless, a suitable mass may be generated by quantum effects. In the absence of supersymmetry, moduli fields tend to develop an effective potential at one-loop order. This happens already in the simplest Kaluza-Klein (KK) compactification, and even if there are no branes (Appelquist and Chodos, 1983a,b). A 5D field χ can be split in an infinite tower of massive KK fields, labeled by a discrete index n. The masses of the KK fields $m_n(\varphi)$ depend on the size φ of the extra dimension, and because of that an effective potential $V(\varphi)$ is generated at one loop. From the 5D perspective, this corresponds to the (nonlocal) Casimir energy density due to the presence of compact directions.

In most cases, the self-gravity of brane and bulk matter content induce a warp in the extra dimension. In this situation, the renormalization of $V(\varphi_i)$ is more elaborate than it is in flat space. In particular, local terms proportional to worldvolume operators on the brane are generated, which may give rise to interesting physical effects. Here, I report on the work done in this subject in collaboration with Oriol Pujolas and Takahiro Tanaka. In Garriga *et al.* (2001a), we considered the particular case of the RS model. A curious result was the absence of Coleman–Weinberg terms in the effective potential. As mentioned above, in the 4D effective description, the fields living in the negative tension brane have masses which are proportional to the radion expectation value. This is reminiscent of the Higgs mechanism in the standard model, and naively one might have expected the usual 4D logarithmic effective potential for the radion $V(\psi) \propto \psi^4 \ln \psi$. However, we showed that a regularization which would preserve the 5D general covariance did not produce such logarithm (see also Brevik *et al.*, 2000; Flachi and Toms, 2001; Goldberger and Rothstein, 2000; Nojiri *et al.*, 2000; Toms, 2000). Initially, this result was somewhat discouraging since it means that the one-loop effective potential induced by gravitons was not useful in the context of a solution to the hierarchy problem. Indeed, the requirement of a large hierarchy required a fine-tuning of parameters (in addition to the usual fine-tuning of the cosmological constant), which in turn resulted in a very small mass for the radion, well below the phenomenologically acceptable values.

Recently, however, we showed in Garriga and Pomarol (2002) that bulk gauge fields (or any of their supersymmetric relatives) can do the job. These fields induce logarithmic contributions to the radion effective potential which are sufficient to stabilize it, generating a large hierarchy of scales without fine-tuning. In this case, the effective potential takes the form $V(\psi) \propto \psi^4/(\ln \psi)$. The logarithmic behavior can be understood, in a 4D holographic description, as the running of gauge couplings with the infrared cutoff scale (which corresponds to the electroweak scale). Thus, Casimir forces due to bulk gauge fields provide the stabilization mechanism which is necessary for a complete solution to the hierarchy problem in the context of the RS model.

The RS model is a somewhat special case, in that the bulk and branes are maximally symmetric, and all possible counterterms amount to renormalizations of the brane tensions. A more interesting behavior of the effective potential is expected in warped brane-world models which are not maximally symmetric, and in this case one may expect that the hierarchy can be generated without recourse to the peculiar behavior of bulk gauge fields.

To clarify this issue, in Garriga *et al.* (2001b) we considered a class of models in which the bulk is no longer AdS. The model contains a bulk "dilaton" scalar field with an exponential potential which is coupled to the 5D Einstein gravity. This provides a family of solutions with two classically massless moduli (a combination of which is the distance between branes). The solutions have a power-law warp factor $a(y) \propto y^q$ instead of the exponential warp of the RS model. Here y is the proper distance in the extra dimension and q is a constant which depends on the parameters in the Lagrangian. For $q = 1/6$, this reduces to the heterotic M-theory brane-world of Lukas *et al.* (1999), which may perhaps be relevant for the recently proposed ekpyrotic universe scenario (Khoury *et al.*, 2001), whereas the RS model is recovered in the limit $q \to \infty$.

In what follows, I shall review the main results of these investigations, with an emphasis on the renormalization of the effective potential. (For details, see Garriga *et al.*, 2001a,b.) In Section 2, the 5D model is introduced. In Section 3 we derive the action for the 4D moduli fields. These are massless at the classical level, owing to a scaling symmetry of the 5D action. In Section 4 we give a brief account of quantum effects by considering the effective potential induced by a conformally invariant field. This case is rather trivial, and it properly illustrates the Casimir interaction between the branes. However, it misses the possibility of interesting local terms induced at one loop. Section 5 deals with the formalism for the calculation of $V(\varphi_i)$ in more general cases, and contains the core of the discussion on renormalization. Possible consequences for the stabilization of the moduli are summarized in Section 6.

2. CLASSICAL DYNAMICS

A simple model where the one-loop effects will play an interesting role is a 5D space–time with a nontrivial background scalar field ϕ, which we shall refer to as the *dilaton*. This is necessary to obtain a departure from the maximally symmetric AdS bulk. The fifth dimension is compactified on a Z_2 orbifold, with two branes at the fixed points of the Z_2 symmetry. The action for the background fields is given by

$$S_{\rm b} = \frac{-1}{\kappa_5}\int d^5x\,\sqrt{-g}\left(\mathcal{R} + \frac{1}{2}(\partial\phi)^2 + \Lambda\, e^{c\phi}\right) \\ -\sigma + \int d^4x\,\sqrt{-g_+}\,e^{c\phi/2} - \sigma_- \int d^4x\sqrt{-g_-}\,e^{c\phi/2}, \tag{1}$$

where $\mathcal{R}$ is the curvature scalar and $\kappa_5 = 16\pi G_5$ (where G_5 is the 5D gravitational coupling constant). We have denoted the induced metrics on the positive and negative tension branes by $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$, respectively. A solution of the equations of motion can be found by making an ansatz where the 4D metric is flat,

$$ds^2 = dy^2 + a^2(y)\eta_{\mu\nu}\,dx^\mu\,dx^\nu, \tag{2}$$

with a x^μ-independent scalar field $\phi = \phi(y)$. The positive and negative branes are placed at $y = y_+$ and y_-, respectively. Under these assumptions, there is a solution for any value of c, given by

$$\phi = -\sqrt{6q}\,\ln(y/y_0), \\ a(y) = (y/y_0)^q, \tag{3}$$

where

$$q = \frac{2}{3c^2}, \qquad y_0 = \sqrt{\frac{3q(1-4q)}{\Lambda}} \tag{4}$$

(constant rescalings of the warp factor are of course allowed, but unless otherwise stated, we shall take the convention that $a(y) = 1$ at $y = y_0$), provided that the tensions are tuned to the values

$$\sigma_{\pm} = \pm \frac{1}{\kappa_5} \sqrt{\frac{48q\Lambda}{1-4q}}. \tag{5}$$

In the absence of the branes, the spacetime (3) contains a singularity at $y = 0$. Since we are considering the range between y_- and y_+, this singularity is of course inocuous. Our spacetime consists of two copies of the slice comprised between y_- and y_+, which are glued together at the branes. Hence, the fifth dimension is topologically an S^1/Z_2 orbifold (see Fig. 1).

For $q = 1/6$, this solution is precisely the M-theory heterotic brane of Lukas *et al*. (1999). On the other hand, the RS case, where the bulk is AdS and there is no scalar field, can be obtained by taking the limit $q \to \infty$ and $y_0 \to \infty$ simultaneously, while its ratio is kept fixed,

$$\ell = \lim_{q\to\infty} \frac{y_0}{q} = \sqrt{\frac{-12}{\Lambda}}. \tag{6}$$

Defining $y \equiv y_0 + y^*$, we find that in the limit the warp factor becomes an exponential

$$\lim_{q\to\infty} a = e^{y^*/\ell},$$

which corresponds to AdS space with curvature radius equal to ℓ.

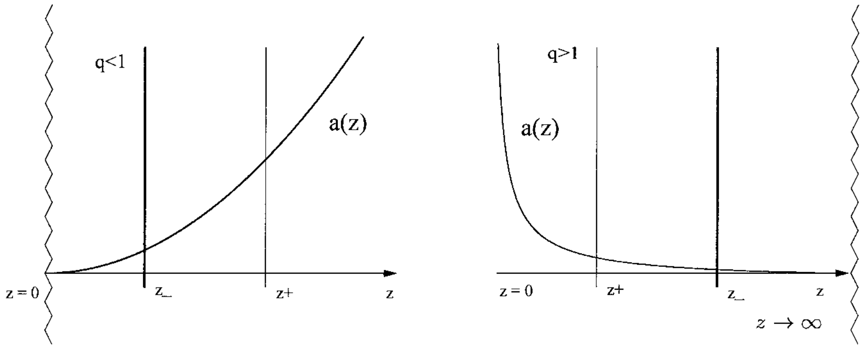

Fig. 1. For $q < 1$, in conformal coordinate, the singularity sits at $z = 0$, and the coordinate of the negative tension brane is smaller than the coordinate for the positive tension one, $\eta = z_+/z_- > 1$. For $q > 1$, the singularity is placed at $z \to \infty$ and z_+, and so $\eta < 1$.

3. MODULI FIELDS

For fixed value of the coupling c, the solution given above contains only two physically meaningful free parameters, which are the locations of the branes y_- and y_+. This leads to the existence of the corresponding moduli, which are massless scalar fields from the 4D point of view. In addition to these moduli, the massless sector also contains the graviton zero mode. To account for it, we generalize our metric ansatz (2) by promoting $\eta_{\mu\nu}$ to an arbitrary 4D metric:

$$ds^2 = dy^2 + a^2(y)\tilde{g}_{\mu\nu}(x)\,dx^\mu\,dx^\nu. \tag{7}$$

The metric induced on the branes is given by

$$g^{\pm}_{\mu\nu} = a^2_{\pm}[\tilde{g}_{\mu\nu} + a^{-2}_{\pm}\partial_\mu y_\pm \partial_\nu y_\pm].$$

Here, and in what follows, the subindices $\pm$ mean that the quantity is evaluated at the *perturbed* brane location.

Substituting the previous expressions into the action (1), with the addition of the extrinsic curvature terms, and using the background equations of motion we find

$$S_{\rm b} = \frac{-1}{16\pi G}\int d^4x\,\sqrt{-\tilde{g}}\left\{(\varphi_+^2 - \varphi_-^2)\tilde{\mathcal{R}} - \frac{6q}{q+1/2}[(\tilde{\partial}\varphi_+)^2 - (\tilde{\partial}\varphi_-)^2]\right\}. \tag{8}$$

Here we have introduced

$$\varphi_\pm \equiv \left(\frac{y_\pm}{y_0}\right)^{q+1/2},$$

and the 4D Newton's constant G given by

$$G = \left(q + \frac{1}{2}\right)\frac{G_5}{y_0}. \tag{9}$$

The modulus corresponding to the positive tension brane has a kinetic term with the “wrong” sign. However, this does not necessarily signal an instability, because it is written in a Brans–Dicke frame. One may go to the Einstein frame by a conformal transformation. It is convenient to introduce the new moduli φ and ψ through (Khoury *et al.*, 2001)

$$\varphi_+ = \varphi\,\cosh\psi, \qquad \varphi_- = \varphi\,\sinh\psi,$$

and to define the new metric

$$\hat{g}_{\mu\nu} = \varphi^2\tilde{g}_{\mu\nu}.$$

It is then straightforward to show that $\varphi^2\sqrt{\tilde{g}}\tilde{\mathcal{R}} = \sqrt{-\hat{g}}[\hat{R} + 6\varphi^{-2}(\hat{\partial\varphi})^2]$. Substituting into the background action (8), we have

$$S_{\mathrm{b}} = \frac{-1}{16\pi G} \int d^4x \sqrt{-\hat{g}} \left\{ \hat{\mathcal{R}} + \frac{6}{1+2q} \frac{(\hat{\partial}\varphi)^2}{\varphi^2} + \frac{12q}{1+2q} (\hat{\partial}\psi)^2 \right\}. \tag{10}$$

Therefore, both moduli have positive kinetic terms in the Einstein frame. At the classical level, the moduli are massless, but as we shall see in the following sections, a potential term of the form

$$\delta S = -\int d^4x \, V(\phi, \psi) \equiv -\int d^4x \sqrt{-\hat{g}} \hat{V}(\varphi, \psi) \tag{11}$$

is generated at one loop, which should be added to (10).

In the RS limit $q \to \infty$ [see Eq. (6)], the kinetic term for one of the moduli disappears. This is to be expected, because the bulk is the maximally symmetric AdS space. In this case only the relative position of the branes, $y_+ - y_-$, is physically meaningful and the other modulus can be gauged away (see also Barvinsky, 2001, for a recent discussion of this case).

Since we are interested in the effective potential for the moduli, it is perhaps pertinent to start by asking why are these fields massless at the classical level. The reason is that under the global transformation

$$g_{ab} \to T^2 g_{ab}, \tag{12}$$

$$\phi \to \phi - (2/c) \ln T, \tag{13}$$

the action (1) scales by a constant factor

$$S_{\mathrm{b}} \to T^3 S_{\mathrm{b}}.$$

Here g_{ab} is the metric appearing in the action (1). Acting on a solution with one brane, the transformation simply moves the brane to a different location. Hence, all brane locations are allowed, from which the masslessness of the moduli follows. However, we should hasten to add that this is just a global scaling symmetry which need not survive quantum corrections.

It is interesting to observe that by means of a conformal transformation, we may construct a new metric $g_{ab}^{(s)}$ which is invariant under the scaling symmetry

$$g_{ab}^{(s)} = e^{c\phi} g_{ab}. \tag{14}$$

Now, the symmetry is a mere shift in ϕ. Moreover, with our background solutions for g_{ab} and ϕ, the metric $g_{ab}^{(s)}$ is just AdS, as can be easily shown from (14) and (3).

4. EFFECTIVE POTENTIAL INDUCED BY BULK CONFORMAL FIELDS

Before embarking on a detailed discussion of renormalization, we shall consider in this section the case of a conformal bulk field χ. This case is rather easy to handle, and it is useful in illustrating the nonlocal contribution of the vacuum energy to the masses of the moduli.

Following the discussion given in Section III of Garriga *et al.* (2001a), we define the conformal coordinates by

$$z \equiv \left| \int \frac{dy}{a(y)} \right| = \frac{y_0}{|1-q|} \left(\frac{y}{y_0} \right)^{1-q}, \tag{15}$$

and we rewrite the metric as

$$ds^2 = a^2(z)(dz^2 + \eta_{\mu\nu}\, dx^\mu\, dx^\nu), \quad a(z) = (z/z_0)^\beta, \tag{16}$$

where

$$\beta = \frac{q}{1-q}, \qquad z_0 = \frac{y_0}{|1-q|}. \tag{17}$$

Here we should mention that the direction of increasing z does not coincide with the direction of increasing y when $q > 1$.

The Casimir energy density ρ of the conformally coupled scalar field in this spacetime is related to its counterpart in a flat spacetime with a compactified extra dimension ($a = 1$ in the above metric) by a conformal transformation. The relation is $\rho = a^{-5}$, where ρ_0 is the flat space value:

$$\rho_0 = \frac{V_0}{2|z_+ - z_-|} = \mp \frac{A}{2|z_+ - z_-|^5}; \quad A \equiv \frac{\pi^2}{32} \zeta_R'(-4) \approx 2.46 \times 10^{-3}. \tag{18}$$

Here z_+ and z_- are the positions of the positive and the negative tension branes in the conformal coordinate z. The double signs in (18) refer to bosons or fermions respectively. Then, we find that the contribution of a conformally coupled scalar field to the effective potential per unit co-moving volume is given by

$$V(z_+, z_-) = \sqrt{-\hat{g}}\hat{V} = 2 \int a^5(z) \rho\, dz = \mp \frac{A}{|z_+ - z_-|^4}. \tag{19}$$

Here, $\hat{V}$ is the effective potential per unit "physical" volume (i.e. the volume as measured with the Einstein metric $\hat{g}$) to be inserted in (10). In terms of the moduli fields φ and ψ, we have

$$\hat{V}(\varphi, \psi) = \mp B \Lambda^2 \varphi^{-12\gamma} [(\cosh \psi)^{3\gamma - 1} - (\sinh \psi)^{3\gamma - 1}]^{-4}, \tag{20}$$

where we have defined

$$B = \frac{A(1-q)^4}{9q^2(1-4q)^2}, \qquad \gamma = \frac{1}{1+2q},$$

and we have used $\sqrt{\hat{g}} = \varphi^4$ for the background solution. It should be mentioned that local terms induced by quantum corrections may be added to (20), both because of fields which live on the branes as well as from nonconformal bulk fields. A discussion of these terms is deferred to the next section.

In the RS case $\gamma \to 0$ the φ dependence in (20) disappears, and we recover the results of Garriga *et al.* (2001a). We have

$$\hat{V}_{\mathrm{RS}} = \pm \frac{A}{16\ell^4} e^{4\psi} \sinh^4 2\psi.$$

For $\psi \ll 1$, the field ψ corresponds to the hierarchy between scales on both branes:

$$\tanh \psi \approx \psi = \frac{a_-}{a_+} \sim \frac{\mathrm{TeV}}{m_{\mathrm{Pl}}} \ll 1,$$

where $m_{\mathrm{Pl}} \sim 10^{19}$ GeV is the Planck mass. In the second equation we assume that the warp is responsible for the hierarchy between Planck and electroweak scales. In this case we have

$$\hat{V}_{\mathrm{RS}}(\psi) \approx \frac{A}{\ell^4} \psi^4 [1 + 4\psi + \cdots] \sim A(\mathrm{TeV})^4.$$

Here we have assumed that the AdS radius is not far below the Planck scale, in which case the contribution to the effective potential is of electroweak order. This huge contribution must somehow be cancelled by some other term in order to have an acceptable 4D cosmological constant. Also, the slope of the potential has to be cancelled at the place where the radion sits. These two conditions can be imposed if we allow for a finite renormalization of the brane tensions, which contribute proportionally to $a_+^4 \propto \cosh^4 \psi \approx 1$ and $a_-^4 \propto \sinh^4 \psi \approx \psi^4$ for the positive and negative tension branes respectively. With these additions we have

$$\hat{V}_{\mathrm{RS}}(\psi) = A\ell^{-4}[c_1 + c_2\psi^4 + 4\psi^5 + \cdots],$$

where c_1 and c_2 are undetermined constants which can only be fixed by experiment. The "renormalization" condition $\partial_\psi V(\psi) = 0$ at the observed value of the radion $\psi = \psi_{\mathrm{obs}} \sim \mathrm{TeV}/m_{\mathrm{Pl}}$ forces $c_2 \sim \psi_{\mathrm{obs}}$. Hence the mass of the radion induced by this quantum correction is given by (Garriga *et al.*, 2001a)

$$m_\psi^2 \approx A \frac{\psi_{\mathrm{obs}}^3}{\ell^4 m_{\mathrm{Pl}}^2} \sim A\psi_{\mathrm{obs}}(\mathrm{TeV})^2. \tag{21}$$

Again, in the last equality, we have taken ℓ to be similar to the Planck scale. The radion mass is much lower than the electroweak scale and hence it is not enough for a phenomenologically viable stabilization of the radion. The reason for such a small mass is that $V(\psi)$ is polynomial for small ψ, with the first power being ψ^4. The situation would be different if the potential took the form $\psi^4 \log(\psi)$ as in the Coleman–Weinberg case. Then, the induced mass m_ψ^2 would be of the same order as the induced potential. Remarkably, the absence of logarithmic terms extends to the contribution from massive fields too (Goldberger and Rothstein, 2000) (see, however, Garriga and Pomarol, 2002, for an exceptional case which may stabilize the radion at an acceptable mass).

For the more general case of a power-law warp factor, the possibility of an efficient stabilization by quantum effects will depend on the nature of the local operators which are induced by quantum corrections. We now turn to a discussion of this subject.

5. EFFECTIVE POTENTIAL IN THE GENERAL CASE

In this section we set up the framework for computing the contribution to the one-loop effective potential from a scalar field χ propagating in the bulk with a generic mass term, which may include couplings to the curvature of spacetime as well as couplings to the background dilaton ϕ. The effective potential for the moduli $y_\pm$ will be defined as usual in terms of a Gaussian path integral around the background solution. Before presenting the actual calculation, however, a digression on the choice of the measure of integration will be useful.

A quantum field theory is defined not just by the classical Lagrangian, but it is also necessary to specify the measure of functional integration. The latter is usually prescribed by demanding certain symmetries or invariances. For instance, for scalar fields in curved space, invariance under diffeomorphisms is an obvious requirement. If gravity were the only background field, then this requirement would suffice to uniquely define Gaussian integration around that background. On the other hand, if there are fields other than gravity with a nontrivial profile (such as our dilaton ϕ), then there is a wide class of possible choices, related to each other by dilaton-dependent conformal transformations. All choices within this class are equally good from the point of view of diffeomorphism invariance. Strictly speaking, however, they are inequivalent becuase of the well-known conformal anomaly.

To be definite, let us concentrate in the simple case of a bulk scalar field χ with canonical kinetic term. The (Euclidean) action for this field is given by

$$S[\chi] = \frac{1}{2} \int d^D x \sqrt{-g} \chi P \chi, \tag{22}$$

where we have introduced the covariant operator

$$P = -(\Box_g + E).$$

Here, $\Box_g$ is the d'Alembertian operator associated with the metric g_{ab}, and $E = E[g_{ab}, \phi]$ is a generic "mass" term. Typically, this takes the form $E = -m^2 - \xi \mathcal{R}_g$, where m is a constant mass, $\mathcal{R}_g$ is the curvature scalar, and ξ is an arbitrary coupling. Throughout this section we shall leave E unspecified.

A volume measure $\mathcal{D}_\chi$ in field space $\mathcal{F}$ can be found from a metric G_{xy} on $\mathcal{F}$, through the relation

$$\mathcal{D}_\chi = \sqrt{G} \prod_x d\chi^x. \tag{23}$$

Here, the spacetime coordinates x and y are considered as continuous labels for the coordinates $\chi^x \equiv \chi(x)$ of the infinite dimensional space $\mathcal{F}$, and G is the determinant of G_{xy}. To specify G_{xy}, we note that a natural definition of a scalar product in the space of field variations $\delta\chi$ can be given in terms of the spacetime measure $d\mu(x)$, through the relation

$$\langle\delta\chi_1, \delta\chi_2\rangle_\mu \equiv \int d\mu(y)\, d\mu(x)\, G_{xy}\delta\chi_1^x\delta\chi_2^y \equiv \int d\mu(x)\, \delta\chi_1(x)\delta\chi_2(x).$$

We denote field variations by $\delta\chi$, just to emphasize that we are refering to elements of the tangent space. More precisely, $\delta\chi = \delta\chi^x\, e_x$, where $e_x = \partial/\partial\chi^x$ is the coordinate basis of the tangent space at the point p, which corresponds to the background solution. In a Riemannian spacetime, the invariant measure is given by

$$d\mu(x) = \sqrt{g(x)}\, d^D x, \tag{24}$$

where g is the determinant of g_{ab}, and D is the dimension. The implicit definition of G_{xy} given above is just the identity $\delta_\mu(x, y)$ with respect to $d\mu$ integration:

$$G_{xy} = \delta_\mu(x, y) = \frac{\delta^{(n)}(x - y)}{\sqrt{g(x)}}. \tag{25}$$

It is convenient to express the field variations in an orthonormal basis χ_n, with $\langle\chi_n, \chi_m\rangle = \delta_{nm}$, so that $\delta_\chi(x) = \sum_n c^n \chi_n(x)$. In this basis, the components of the field variation are c^n, and the metric is just the usual delta function (the continuous or the discrete delta function, depending on whether the normalization of χ_n, is continuous or discrete):

$$G_{nm} = \delta_{nm}.$$

Substituting this in (23), we have

$$\mathcal{D}_\chi = \prod_n dc^n. \tag{26}$$

It should be clear from the previous discussion that the definition of $\mathcal{D}_\chi$ is associated with a natural definition of $d\mu(x)$. However, in the problem under consideration, the choice of $d\mu$ is not unique. In our case, there is a nontrivial dilaton field ϕ, and we can consider a whole class of spacetime measures of the form

$$d\mu\theta(x) = \sqrt{g_\theta}\, d^D x = \Omega_\theta^D(\phi)\sqrt{g}\, d^D x,$$

which correspond to conformally related metrics

$$g_{ab}^\theta = \Omega_\theta^2 g_{ab},$$

for an arbitrary function $\Omega_\theta(\phi)$. In the presence of a dilaton, the coupling to gravity is not universal and it is not clear which one of these metrics should be considered more physical. To proceed, it is convenient to define the operator P_θ associated with the metric g^θ_{ab} by

$$\Omega_\theta^{(D-2)/2} P_\theta \Omega_\theta^{(2-D)/2} = \Omega_\theta^{-2} P, \tag{27}$$

where P was introduced after Eq. (22). This operator can be written in covariant form as

$$P_\theta = -(\Box_\theta + E_\theta),$$

where

$$E_\theta = \left(\frac{D-2}{2}\right) \Box_\theta \ \ln \Omega_\theta - \left(\frac{D-2}{2}\right)^2 g_\theta^{ab} \partial_a \ \ln \Omega_\theta \ \partial_b \ \ln \Omega_\theta + \Omega_\theta^{-2} E,$$

and $\Box_\theta$ is the covariant d'Alembertian corresponding to g^θ_{ab}. Introducing $\chi_\theta \equiv \Omega^{(2-D)/2}\chi$, the action for the scalar field can be expressed as

$$S[\chi] = \frac{1}{2} \int d^D x \sqrt{g_\theta}\, \chi_\theta P_\theta \chi_\theta. \tag{28}$$

In terms of g^θ_{ab} the field χ_θ has a perfectly canonical and covariant kinetic term.

Thus, the same arguments which lead to (25) can now be used to find the natural line element in field space associated with the spacetime measure $d\mu_\theta(x)$. In the basis $\{\chi_{\theta n}\}$, which is orthonormal with respect $d\mu_\theta$, the field variation can be expanded as $\delta\chi_\theta(x) = \sum_n c^n_\theta \chi_{\theta n}$, and the new measure takes the form

$$(\mathcal{D}\chi)_\theta = \prod_n dc^n_\theta. \tag{29}$$

Using $\chi_{\theta m} = \Omega_\theta^{-D/2}\chi_m$, it is straigtforward to show that $c^m = M^m_n c^n_\theta$, where $M^m_n = \langle \chi_m, \Omega_\theta^{-1}\chi_n\rangle_\mu \equiv (\Omega_\theta^{-1})^m_n$. Hence the two measures (26) and (29) are related related by

$$\mathcal{D}\chi = J_\theta (\mathcal{D}\chi)_\theta, \tag{30}$$

where the Jacobian is formally given by

$$J_\theta = \det\left(\Omega_\theta^{-1}\right) = \exp[-\mathrm{Tr}\ \ln \Omega_\theta]. \tag{31}$$

In the last step we have used the formal definition of the L_2 trace[2]:

$$\mathrm{Tr}[\mathcal{O}] = \sum_m \int d^D x\, g^{1/2} \chi_m(\mathcal{O}\chi_m) = \sum_m \int d^D x\, g_\theta^{1/2} \chi_{\theta m}(\mathcal{O}\chi_{\theta m}).$$

[2] The definition of the trace is robust, in the sense that it is independent of the metric one uses in order to define the orthonormal basis, as long as the corresponding measures are in the same L_2 class. This will be the case, for instance, if the metrics are related by a conformal factor which is bounded above and below on the manifold.

The trace is well defined if the diagonal matrix elements of the operator $\mathcal{O}$ decay sufficiently fast at large momenta. Unfortunately, the diagonal matrix elements of $\ln \Omega_\theta$ do not decay at all at large m, and so the trace is ill-defined unless we introduce a regulator. We will address this question below, where we will explicitly define what we mean by J_θ.

Since we have a classical scaling symmetry in the gravity and dilaton sector, one could argue that $g^{(s)}_{ab}$, which is invariant under scaling (see Section 3), is the preferred physical metric. However, even in this case the divergent part of the effective potential will not respect the scaling symmetry, and consequently we need to introduce counterterms with the "wrong" scaling behavior. Hence, in what follows, we shall take the conservative attitude that the measure is determined in the context of a more fundamental theory (from which our 5D effective action is derived). and we shall formally consider on equal footing all choices associated with metrics in the conformal class of g_{ab}, including of course $g^{(s)}_{ab}$. As we shall see, the difference between these choices amounts to the addition of local terms in the effective potential.

The contribution of the field χ to the renormalized effective potential V_θ per unit co-moving volume parallel to the branes is given by

$$\exp\left[-\mathcal{A}\big(V_\theta + V_\theta^{\text{div}}\big)\right] \equiv \int (\mathcal{D}\chi)_\theta \, e^{-S[\chi]} = (\det\ P_\theta)^{-1/2}, \tag{32}$$

where $\mathcal{A}$ is the co-moving volume under consideration and we have used (28) and the measure (29) to express the Gaussian integral as a determinant. The term V_θ^{div} is a local counterterm, which, in dimensional regularization, needs to be subtracted from the regularized effective potential (its explicit form will be given in the coming sections). In zeta function regularization, the left-hand side is already finite, and V_θ^{div} is unnecessary (it corresponds to a finite renormalization of couplings). Equation (32) can be written as

$$V_\theta = \frac{1}{2\mathcal{A}}\ \ln(\det P_\theta) - V_\theta^{\text{div}}. \tag{33}$$

Equation (30) suggests the notation

$$V_\theta = \frac{1}{2\mathcal{A}}\ \ln(\det P) - V_\theta^{\text{div}} + \frac{1}{\mathcal{A}}\ \ln J_\theta. \tag{34}$$

The reader should be aware, however, that the definition of the Jacobian in Eq. (31) is only formal because the trace in the right-hand side of this equation is ill-defined. For that reason, it is not clear that Eq. (34) would hold with the definition (31), after substituting determinants by traces and applying any kind of regularization to the formally divergent traces. To avoid misinterpretations, in the discussions that follow we shall take J_θ to be defined by Eq. (34), that is

$$\ln\ J_\theta \equiv \frac{1}{2}[\ln(\det P_\theta) - \ln(\det P)], \tag{35}$$

where the expression in the right-hand side is to be calculated in some regularization scheme.

The way the θ dependence of V_θ arises is very different in different regularization schemes. In Eq. (34), the determinant of P is independent of θ (we recall that this operator corresponds to the choice $\Omega_\theta = 1$). In dimensional regularization, ln J_θ vanishes, but the divergent term $V_\theta^{\rm div}$, which is subtracted from ln(det P), depends on the choice of physical metric g^θ_{ab}. On the other hand, in zeta function regularization, ln(det P), is finite and $V^{\rm div}$ does not play a role (in any case, any finite renormalization does not introduce a dependence in θ). Rather, in this case, the dependence on θ comes from ln J_θ, which does not vanish in this regularization scheme. In both cases, the θ dependence of V_θ is the same.

As we shall see, this dependence can be cast in the form of local operators on the branes, and therefore the ambiguity in the choice of the integration measure can also be understood as modification of the classical action. It should be noted, however, that the local operators which result from a shift in θ have different form than the terms arising from the usual shift in the renormalization constant μ, which inevitably crops up in the regularized traces. In the cases we shall consider, the latter will take the form $K^4(y_\pm)$, where K denotes terms which behave like the extrinsic curvature of the branes at the positions $y_\pm$. On the other hand, the θ-dependent terms behave as $K^4(y_\pm)\phi(y_\pm)$. Since $K(y)$ behaves like the inverse of y whereas $\phi(y)$ behaves logarithmically with y, these terms will give rise to Coleman–Weinberg-type potentials for the moduli.

5.1. Conformal Transformations and the KK Spectrum

The direct evaluation of the determinant of P_θ appearing in Eq. (32) turns out to be rather impractical, because of the complicated form of the implicit equation which defines its eigenvalues. For actual calculations it is convenient to work with a conformally related operator P_0 whose eigenvalues will be related to the KK masses.

Following Garriga *et al.* (2001a), we introduce a one-parameter family of metrics which interpolate between a fictitious flat spacetime and any of the metrics in the conformal class of the Einstein metric:

$$g^\theta_{ab} = \Omega^2_\theta(\phi) g_{ab}, \tag{36}$$

where θ parametrizes a path in the space of conformal factors. For definiteness we shall restrict attention to conformal factors $\Omega_\theta(\phi)$, which have an exponential dependence on the dilaton:

$$\Omega_\theta(z) = e^{(1-\theta)\phi/3c} = \left(\frac{z}{z_0}\right)^{\beta(\theta-1)}. \tag{37}$$

With this choice, $\theta = 0$ represents flat space and $\theta = 1$ corresponds to the Einstein frame metric (16). For $\theta = -1/\beta$, the metric g^θ_{ab} coincides with the metric $g^{(s)}_{ab}$

introduced in Section 3, which is invariant under the scaling transformation (as mentioned before, this metric corresponds to a 5D AdS space, with curvature radius given by z_0).

The operator $P_0 \equiv P_{\theta=0}$ is the wave operator for the KK modes, which one would use in a 4D description. The Lorentzian equation of motion $P\chi = 0$ can be written as

$$P_0 \chi_0 = 0,$$

where

$$P_0 = -\Box_{D-1} + \hat{M}^2(z). \tag{38}$$

Here, $\Box_{D-1}$ is the flat space d'Alembertian along the branes, and

$$\hat{M}^2 \equiv -\partial_z^2 - E_0$$

is the Schrödinger operator whose eigenvalues are commonly referred to as the KK masses m_n:

$$\hat{M}^2(z)\chi_{0,n}(z) = m_n^2 \chi_{0,n}(z). \tag{39}$$

The interesting feature of (38) is that it separates into a 4D part and a z-dependent part. A mode of the form $\chi_0 = e^{ik_\mu x^\mu} \chi_{0,n}$ will solve the equation of motion (38), provided that the dispersion relation

$$k_\mu k^\mu + m_n^2 = 0$$

is satisfied, and hence modes labeled by n behave as 4D massive particles. Technically, the advantage of working with P_0 is that its (Euclidean) eigenvalues $\lambda_{n,k} = k_\mu k^\mu + m_n^2$ separate as a sum of a 4D part plus the eigenvalue of the Schrödinger problem in the fifth direction.

In the following subsections, we shall discuss how det P_0 is related to the determinant of our interest, det P, or more generally to det P_θ. For completeness, and in order to illustrate practical methods for calculating the effective potential, we shall consider dimensional regularization and zeta function regularization. Both methods will lead to identical results.

5.2. Dimensional Regularization

A naive reduction to flat 4D space suggests that the effective potential can be obtained as a sum over the KK tower:

$$V^D \equiv \mu^\epsilon \sum_n \frac{1}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \log\left(\frac{k^2 + m_n^2(\varphi_i, D)}{\mu^2}\right). \tag{40}$$

Here $D = 4 + 1 - \epsilon$ is the dimension of spacetime, and we have added $(-\epsilon)$ dimensions parallel to the brane. The renormalized effective potential should then be given by an expression of the form

$$V(\varphi_i) = V^D - V^{\mathrm{div}}, \tag{41}$$

and the question is what to use for the divergent subtraction V^{div}. Since Eq. (40) is similar to an ordinary effective potential in 4D flat space,[3] one might imagine that V can be obtained from V^D just by dropping the pole term, proportional to $1/\epsilon$; but this is not true for warped compactifications

$$V(\varphi_i) \neq V^D - (\text{pole term}).$$

The point is that the theory is 5D and the spacetime is curved, and this fact must be taken into account in the process of renormalization.

Rather than proceeding heuristically from (41), we must take the definition of the effective potential (33) as our starting point, where it is understood that the formally divergent trace must be regularized and renormalized. In order to identify the divergent quantity to be subtracted, we shall use standard heat kernel expansion techniques. Let us introduce the dimensionally regularized expressions (Buchbinder *et al.*, 1992; Elizalde *et al.*, 1994; Hawking, 1977)

$$V_\theta^D \equiv \frac{\mu^\epsilon}{2\mathcal{A}} \operatorname{Tr} \ln\left(\frac{P_\theta(D)}{\mu^2}\right) = -\frac{\mu^\epsilon}{2\mathcal{A}} \lim_{s\to 0} \partial_s \zeta_\theta(s, D), \tag{42}$$

where

$$\zeta_\theta(s, D) = \operatorname{Tr}\left[\left(\frac{P_\theta(D)}{\mu^2}\right)^{-s}\right] = \frac{2\mu^{2s}}{\Gamma(s)} \int_0^\infty \frac{d\xi}{\xi}\, \xi^{2s} \operatorname{Tr}\big[e^{-\xi^2 P_\theta(D)}\big]. \tag{43}$$

It should be noted that the operator P_θ is positive and therefore the integral is well behaved at large ξ.

As is well known, the regularized potential V_θ^D contains a pole divergence in the limit $D \to 5$. To see that this is the case, one introduces the asymptotic expansion of the trace for small ξ (Branson and Gilky, 1990; De Witt, 1975; McKean and Singer, 1967),

$$\operatorname{Tr}\big[f\, e^{-\xi^2 P_\theta(D)}\big] \sim \sum_{n=0}^{\infty} \xi^{n-D} a_{n/2}^D(f, P_\theta), \tag{44}$$

where $a_{n/2}^D$ are the so-called generalized Seeley–De Witt coefficients. In (44) we have introduced the arbitrary smearing function $f(x)$. This is unnecessary for the present discussion, but it will be useful later on. For $n \leq 5$, their explicit form is known for a wide class of covariant operators, which includes our P_θ. They are finite and can be constructed from local invariants (terms constructed from the

[3] It should also be mentioned that each KK contribution in Eq. (40) is not just like a flat space contribution, because in warped compactifications the KK masses $m_n(\varphi, D)$ depend on the number of external dimensions parallel to the brane.

metric, the mass term E_θ and the smearing function f), integrated over spacetime. For even n, they receive contributions from the bulk and from the branes, whereas for odd n they are made out of invariants on the boundary branes only.

For definiteness, let us focus on the simplest case of a Dirichlet scalar field, satisfying

$$\chi(z_\pm) = 0. \tag{45}$$

We can use the result found in Bordag *et al.* (1996), Kirsten (1998, 2001), Moss and Dowker (1989), Vassilevich (1995) to compute the Seeley–De Witt coefficients for a Dirichlet field with a bulk operator $P = -(\Box + E)$. The lowest order ones for odd n are given by

$$a^D_{1/2}(f, P) = \frac{-(4\pi)^{(1-D)/2}}{4} \sum_{i=\pm} \int_{yi} \sqrt{gi}\, f(x)\, d^{D-1}x, \tag{46}$$

$$\begin{aligned} a^D_{3/2}(f, P) = \frac{-(4\pi)^{(1-D)/2}}{384} \sum_{i=\pm} \int_{yi} \sqrt{gi}\, d^{D-1}x\{ f(96E - 16\mathcal{R} + 8\mathcal{R}_{yy} \\ + 7\mathcal{K}^2 - 10\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}) + O(f_{;y},\ f_{;yy})\}. \end{aligned} \tag{47}$$

The most relevant for our purposes will be $a^D_{5/2}$:

$$\begin{aligned} a^D_{5/2}(f, P) = \frac{-(4\pi)^{(1-D)/2}}{5760} \sum_{i=\pm} \int_{yi} \sqrt{gi} d^{D-1}x \Bigg\{ f \Bigg(720E^2 - 450\mathcal{K}E_{;y} \\ + 360E_{;yy} + 15(7\mathcal{K}^2 - 10\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + 8\mathcal{R}_{yy} - 16\mathcal{R})E \\ + 20\mathcal{R}^2 - 48\Box\mathcal{R} - 17\mathcal{R}^2_{yy} - 8\mathcal{R}_{ab}\mathcal{R}^{ab} + 8\mathcal{R}_{abcd}\mathcal{R}^{abcd} \\ - 20\mathcal{R}_{yy}\mathcal{R} + 16\mathcal{R}_{yy}\mathcal{R} - 10\mathcal{R}_{yy}\mathcal{R}_{yy} - 12\mathcal{R}_{;yy} - 15\mathcal{R}_{yy;yy} \\ - 16\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu\rho}\mathcal{R}^{\mu}_{\rho} - 32\mathcal{K}^{\mu\nu}\mathcal{K}^{\rho\sigma}\mathcal{R}_{\mu\rho\nu\sigma} + \frac{215}{8}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\mathcal{R}_{yy} \\ - 25\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\mathcal{R} + \frac{47}{2}\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu}_{\rho}\mathcal{R}^{\rho}_{y\mu y} + \frac{215}{16}\mathcal{R}_{yy}\mathcal{K}^2 \\ - \frac{35}{2}\mathcal{R}\mathcal{K}^2 - 14\mathcal{K}_{\mu\nu}\mathcal{R}^{\mu\nu}\mathcal{K} - \frac{49}{4}\mathcal{K}^{\mu\nu}\mathcal{R}_{\mu y\nu y}\mathcal{K} + 42\mathcal{R}_{;y}\mathcal{K} \\ - \frac{65}{128}\mathcal{K}^4 - \frac{141}{32}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\mathcal{K}^2 + \frac{17}{2}\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu\rho}\mathcal{K}^{\mu}_{\rho}\mathcal{K} \\ + \frac{777}{32}(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu})^2 - \frac{327}{8}\mathcal{K}_{\mu\nu}\mathcal{K}^{\nu\rho}\mathcal{K}_{\rho\sigma}\mathcal{K}^{\mu\sigma}\Bigg) \\ + O(f_{;y}, \ldots, f_{;yyyy})\Bigg\} \end{aligned} \tag{48}$$

Our notation is as follows: E is a general scalar function, $\mathcal{R}^a_{bcd} = +\Gamma^a_{bc,d} - \cdots$ is the Riemann tensor, $\mathcal{R}_{bc} = \mathcal{R}^a_{bac}$ is the Ricci tensor, and $\mathcal{R} = \mathcal{R}_{ab} g^{ab}$ is the curvature scalar. The extrinsic curvature is given by $\mathcal{K}_{\mu\nu} \equiv (1/2)\partial_y g_{\mu\nu}$, where $g_{\mu\nu}(y)$ is the induced metric on y-constant hypersurfaces, and $\mathcal{K} = \mathcal{K}_{\mu\nu} g^{\mu\nu}$. The vector normal to the boundary is ∂_y, and so the normal components are simply the y components. The $a, b, \cdots$ indices run over the extra coordinate, and over the directions tangential to the branes, $\mu, \nu, \cdots$. The omitted terms, represented by $O(f_{;y}, \ldots)$, are linear combinations of the derivatives of f with coefficients which depend on $\mathcal{K}_{\mu\nu}$, E, and its derivatives.

As mentioned above, the integral (43) is well behaved for large ξ. For small ξ, the integral is convergent for $2s > D$, as can be seen from the asymptotic expansion (44). In the end, we have to consider the limit $s \to 0$, and so we must keep track of divergences which may arise in this limit. For this purpose, it is convenient to separate the integral into a small ξ region, with $\xi < \Lambda$, and a large ξ region with $\xi > \Lambda$, where Λ is some arbitrary cutoff. Substituting (44) into (43), we can explicitly perform the integration in the small ξ region for $2s > D$. This gives

$$\zeta(s, D) \sim 2\frac{\mu^{2s}}{\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{\Lambda^{n-D+2s}}{n - D + 2s} a^D_{n/2}(P_\theta) + \int_\Lambda^\infty \frac{d\xi}{\xi}\, \xi^{2s}\, \mathrm{Tr}\left[e^{-\xi^2 P_\theta(D)}\right] \right\}, \tag{49}$$

where we have used the standard notation

$$a^D_{n/2}(P_\theta) = a^D_{n/2}(f = 1, P_\theta).$$

The second term in curly braces is perfectly finite for all values of s. Analytically continuing and taking the derivative with respect to s at $s = 0$, we have

$$\zeta'(0, D) \sim \sum_{n=0}^{\infty} \frac{2\Lambda^{n-D}}{n - D} a^D_{n/2}(P_\theta) + \text{finite}, \tag{50}$$

where the last term is just twice the integral in (49) evaluated at $s = 0$. Introducing the regulator $\epsilon = 5 - D$, the ultraviolet divergent part of V^D_θ is thus given by

$$V^{\text{div}}_\theta = -\frac{1}{\epsilon \mathcal{A}} a^D_{5/2}(P_\theta). \tag{51}$$

The divergence is removed by renormalizing the couplings in front of the invariants which make up the coefficient $a^D_{5/2}$, and so this infinite term can be dropped. The renormalized effective potential of our interest is therefore given by

$$V_\theta = \lim_{D\to 5} \left[V^D_\theta - V^{\text{div}}_\theta\right]. \tag{52}$$

To proceed, we need to calculate V^D_θ, which in principle requires calculating a trace which involves the eigenvalues of P_θ, and as mentioned above, these are not related in any simple way to the KK masses.

However, it turns out that the dimensionally regularized V_θ^D is independent of θ when D is not an integer. The dependence of V_θ^D on θ can be found in the following way. First we note that

$$\partial_\theta \,\mathrm{Tr}\big[e^{-\xi^2 P_\theta}\big] = \mathrm{Tr}\big[2\xi^2 f_\theta(x)\Omega_\theta^{-2} P\, e^{-\xi^2 P_\theta}\big] = -\xi\, \partial_\xi\, \mathrm{Tr}\big[f_\theta(x) e^{-\xi^2 P_\theta}\big], \tag{53}$$

where we have introduced

$$f_\theta \equiv \partial_\theta \,\ln \Omega_\theta,$$

and the cyclic property of the trace was used. The above relation enables us to find the dependence of V_θ^D on the conformal factor:

$$\partial_\theta \lim_{s\to 0} \partial_s \zeta_\theta(s, D) = \lim_{s\to 0} \partial_s \frac{2\mu^{2s}}{\Gamma(s)} \int_0^\infty d\xi\, \xi^{2s} \partial_\xi\, \mathrm{Tr}\big[-\, f_\theta\, e^{-\xi^2 P_\theta}\big]. \tag{54}$$

As with the expansion (49) we may again introduce the regulator Λ and separate the integral into a large ξ part with $\xi > \Lambda$, which is finite, and a small ξ part with $\xi < \Lambda$, which contains the divergent ultraviolet behavior. Assuming that $2s > D$ and integrating by parts, the resulting integrals in the small ξ region can be performed explicitly and we have

$$\partial_\theta \lim_{s\to 0} \partial_s \zeta_\theta(s, D) \sim \lim_{s\to 0} \partial_s \frac{4s\mu^{2s}}{\Gamma(s)} \left[\sum_{n=0}^{\infty} \frac{\Lambda^{n-D+2s}}{n-D+2s} a_{n/2}^D(f_\theta, P_\theta) + \text{finite}\right]. \tag{55}$$

As before, the last term just indicates the integral in the large ξ region. Provided that D is not an integer, all terms in square brackets remain finite at small s, and so the right-hand side of (55) vanishes. Hence, we find that

$$\partial_\theta V_\theta^D = 0 \quad (D \neq \text{integer}). \tag{56}$$

In other words, the dimensionally regularized determinant of P_θ coincides with the dimensionally regularized determinant of P_0, and we have

$$\begin{aligned} V_\theta^{\mathrm{D}} &= V_0^D \equiv V^D \\ &\equiv \sum_n \mu^\epsilon \frac{1}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \log\left(\frac{k^2 + m_n^2(\varphi_i, D)}{\mu^2}\right) (D \neq \text{integer}). \end{aligned} \tag{57}$$

As was anticipated, we find that $\ln J_\theta$ vanishes in the dimensional regularization presented here in the sense given in (35).

Finally, from (52) and (57), the renormalized effective potential is given by

$$V_\theta(\varphi) = \lim_{D\to 5} \left[V^D - \frac{1}{(D-5)} \frac{1}{\mathcal{A}} a_{5/2}^D(P_\theta)\right], \tag{58}$$

where the Seeley–De Witt coefficient $a^D_{5/2}$ is given in (48) with $f = 1$. The above equation bears the ambiguity in the choice of integration measure in the second term in square brackets. Different values of θ give different results. If we take g_{ab} as the preferred metric, then we should use $\theta = 1$, whereas if we take $g^{(s)}_{ab}$ as the preferred metric, we should use $\theta = -1/\beta$. As we shall see in the next subsection, when we set $D = 5$ the coefficient $a_{5/2}(P_\theta)$ is also independent of θ. Hence, the pole term in the second term in (58) is independent of θ, as it should, in order to cancel the pole in V^D. However, the finite part does depend on the choice of θ.

The right-hand side of (58) is ready for explicit evaluation. For instance, the case of massless fields with arbitrary coupling to the curvature,

$$E = -\xi \mathcal{R}_g,$$

and with Dirichlet boundary conditions has been discussed in Garriga *et al.* (2001b). The result is the following: In the limit of small separation between the branes, $(1 - \tau) \ll 1$, the integral $\mathcal{V}$ behaves like $(1 - \tau)^{-4}$, and the logarithmic terms can be neglected. In this limit, the potential behaves like the one for the conformally coupled case, given in (19):

$$V_\theta(z_+, z_-) \sim -\frac{A}{|z_+ - z_-|^4}. \tag{59}$$

For $\tau \ll 1$, when the branes are well separated, the integral $\mathcal{V}$ behaves like $\tau^{2\nu}$ and becomes negligible in the limit of small τ (except in the special case when ν is very close to 0). In this case, we have

$$V_\theta(z_+, z_-) \sim \frac{\beta_4(\beta\theta + 1)}{(4\pi)^2}\left[\frac{\ln(\mu_1 z_+)}{z_+^4} + \frac{\ln(\mu_2 z_-)}{z_-^4}\right] + O\left[\left(\frac{z_<}{z_>}\right)^{2\nu}\right]. \tag{60}$$

Because of the presence of the logarithmic terms, it is in principle possible to adjust the parameters μ_1, μ_2 so that there are convenient extrema for the moduli z_+ and z_-.

5.3. Zeta Function Regularization

The method of zeta function regularization exploits the fact that the formal expression for the effective potential (42) is finite if the limit $D \to 5$ is taken before the limit $s \to 0$. This can be seen from Eq. (49), where the term with $n = 5$ is finite if we set $D = 5$ before taking the derivative with respect to s and setting $s \to 0$. Clearly, the change in the order of the limits simply removes the divergent term V^{div} given in (51) and it reproduces the results obtained by the method of dimensional regularization (up to finite renormalization terms which are proportional to the geometric invariant $a^{D=5}_{5/2}(P_\theta)$).

In zeta function regularization we define

$$V_\theta \equiv -\frac{1}{2\mathcal{A}} \ln(\det P_\theta) \equiv -\frac{1}{2\mathcal{A}} \lim_{s\to 0} \partial_s \zeta_\theta(s), \tag{61}$$

where $\zeta_\theta(s) \equiv \zeta_\theta(s, D = 5)$ [see Eq. (43)]. As in the case of dimensional regularization, it is more convenient to calculate V_0 than V_θ since the eigenvalues of P_0 are related to the spectrum of KK masses. An important difference with dimensional regularization is that

$$-2\mathcal{A}\partial_\theta V_\theta = \partial_\theta \zeta'_\theta(0) = 2a_{5/2}(f_\theta, P_\theta) \neq 0,$$

a result which we already encountered in Garriga *et al.* (2001a) (see also Bordag *et al.*, 1996; Kirsten, 1998, 2001). This can be seen from (55). If we set $D = 5$ from the very beginning, the term with $n = 5$ in Eq. (55) is linear in s, and its derivative with respect to s does not vanish in the limit $s \to 0$. Here, and in what follows, we use the notation

$$a_{n/2} \equiv \lim_{D\to 5} a^D_{n/2}.$$

Integrating along the conformal path parameterized by θ, we can relate the effective potential per unit co-moving volume V_θ, with the "flat space" effective potential V_0 as

$$V_\theta = V_0 - \frac{1}{\mathcal{A}} \int_0^\theta d\theta'\, a_{5/2}(f_{\theta'}, P_{\theta'}). \tag{62}$$

The general expression for $a_{5/2}(f_\theta, P_\theta)$, which applies to our case, has been derived by Kirsten (1998, 2001). In Garriga *et al.* (2001a) we evaluated the integral in (62) for the RS case, in order to obtain V_1 from V_0. Here we shall present an alternative expression for this integral, which does not require the knowledge of $a_{5/2}(f_\theta, P_\theta)$, but only the knowledge of $a^D_{5/2}(P_\theta)$ for dimension $D = 5 - \epsilon$. This will also illustrate the relation between the method of zeta function regularization and the method of dimensional regularization.

From the asymptotic expansion of the first and the last terms in Eq. (53), we have (Branson and Gilkey, 1994; Dowker and Kennedy, 1978)

$$\partial_\theta a^D_{n/2}(P_\theta) = (D - n) a^D_{n/2}(f_\theta, P_\theta). \tag{63}$$

Integrating over θ, we get

$$(D - 5) \int_0^\theta a^D_{5/2}(f_{\theta'}, P_{\theta'})\, d\theta' = a^D_{5/2}(P_\theta) - a^D_{5/2}(P_0). \tag{64}$$

Writing $D = 5 - \epsilon$, we have

$$V_\theta - V_0 = -\frac{1}{\mathcal{A}} \int_0^\theta a_{5/2}(f_{\theta'}, P_{\theta'})\, d\theta' = \frac{1}{\epsilon\mathcal{A}} \left[a^D_{5/2}(P_\theta) - a^D_{5/2}(P_0)\right]. \tag{65}$$

Note that from (51) and (58), the previous equation can also be written as

$$V^D \equiv V_\theta + V_\theta^{\mathrm{div}} = V_0 + V_0^{\mathrm{div}}. \tag{66}$$

This equation simply expresses the fact that the dimensionally regularized V^D is independent of the conformal parameter θ, as we had shown in the previous subsection [see, e.g., Eq. (56)].

From (63), with $D = n = 5$, one finds that the coefficient $a_{5/2}(P_\theta)$ is conformally invariant (Branson and Gilkey, 1994; Dowker and Kennedy, 1978) and therefore

$$a_{5/2}(P_\theta) = a_{5/2}(P_0). \tag{67}$$

Substituting this into (65), we obtain

$$\int_0^\theta a_{5/2}(f_{\theta'}, P_{\theta'})\, d\theta' = \frac{d}{dD} a^D_{5/2}(P_\theta)\bigg|_{D=5} - \frac{d}{dD} a^D_{5/2}(P_0)\bigg|_{D=5}. \tag{68}$$

Thus, the integral in (62) can be evaluated in two different ways. One is by using the explicit expression of $a_{5/2}(f, P)$ given by Kirsten (1998, 2001). The other is by taking the derivative of the coefficients $a^D_{5/2}(P_\theta)$ given in (48) with $f = 1$, with respect to the dimension. [Note that the terms which are linear in derivatives of f, which we have just indicated symbolically in (48), disappear when f is a constant.]

The quantity discussed above is nothing but the θ dependence induced by the choice of integration measure. In the sense of (35), we have

$$\ln J_\theta = \frac{1}{3c\mathcal{A}} \int_\theta^1 d\theta'\, a_{5/2}(\theta, P_{\theta'}), \tag{69}$$

where we have used $f_\theta = \partial_\theta \ln \Omega_\theta = (1-\theta)\phi/3c$. Clearly, the effect of this factor is just adding local terms expressed solely in terms of ϕ and the metric to the classical action. The dependence of these terms is different from the change which results from a rescaling of the renormalization parameter μ. This corresponds to a shift in the coefficient of local terms proportional to $a_{5/2}(P)$.

6. SUMMARY AND CONCLUSIONS

We have studied a class of warped brane-world compactifications, with a power-law warp factor of the form $a(y) = (y/y_0)^q$ and a dilaton with profile $\phi \propto \ln(y/y_0)$. Here y is the proper distance in the extra dimension. In general, there are two different moduli $y_\pm$ corresponding to the location of the branes. (In the RS limit, $q \to \infty$, a combination of these moduli becomes pure gauge.)

Classically, the moduli are massless, but they develop an effective potential at one loop. We have presented methods for calculating this effective potential, using both zeta function and in dimensional regularization. An important point is that the divergent term to be subtracted from the dimensionally regularized effective potential is proportional to the the Seeley–De Witt coefficient $a_{5/2}$, given in (48). In the RS model, this coefficient behaves much like a renormalization of the brane tension, but it behaves very differently in the general case.

In the limit when the branes are very close to each other, it behaves like $V \propto a^4|y_+ - y_-|^{-4}$, corresponding to the usual Casimir interaction in flat space. Perhaps more interesting is the moduli dependence due to local operators induced on the branes, which are the dominant terms in $V(y_+, y_-)$ when the branes are widely separated. Such operators break a scaling symmetry of the classical action, which we discussed in Section 3, but nevertheless are needed in order to cancel the divergences in the effective potential. If we denote by $K(yi) = q/y_i$ the extrinsic curvature of the brane at the location $y = y_i (i = \pm)$, a renormalization of the brane tension parameters $\sigma_\pm$ in the classical action (1) induces terms proportional to $a(y_i)^4 K_i$ in the effective potential. These terms scale like the rest of the classical action under the global transformation (12)–(13). On the other hand, the divergences in the effective potential, proportional to the coefficient $a_{5/2}(1, P)$, require worldsheet counterterms which are proportional to $a(y_i)^4 K^4(yi)$. These have the wrong scaling behavior [they simply do not change under (12)–(13)] and hence they act as stabilizers for the moduli (Garriga *et al.*, 2001b).

In addition, there are terms proportional to $a_{5/2}(\phi, P)$, which contains combinations of the form $a(y_i)^4 K^4(y_i)\phi(y_i)$. The coefficient in front of the latter terms depends on the choice of the measure in the path integral. Different choices are possible, which are related among each other by dilaton-dependent conformal transformations. Because of the conformal anomaly, different choices are inequivalent, but they are simply related by the addition of worldsheet operators proportional to $a_{5/2}(\phi, P)$. Since ϕ behaves logarithmically, these terms have the form of Coleman–Weinberg-type potentials for the moduli y_i, and they can also act as stabilizers for the moduli.

As shown in Garriga *et al.* (2001b), worldsheet operators induced on the brane at one loop easily stabilize the moduli in brane-world scenarios with warped compactifications, and give them sizable masses. If the warp factor is in the range, $9 \gtrsim 10$, then this stabilization naturally generates a large hierarchy. In this case, the mass of the lightest modulus is somewhat below the TeV scale.

Aside from the local worldsheet operators, which have been the main focus of the present discussion, the nonlocal Casimir interaction due to bulk fields may also be relevant. This is particularly important in the RS case, where all worldsheet operators are proportional to the brane tensions and the above-mentioned mechanism does not furnish a suitable stabilization. As shown in Garriga and Pomarol (2002), the Casimir energy due to bulk gauge fields (or any of their supersymmetric relatives) can in this case stabilize the radion at a hierarchical distance without any fine-tuning of parameters in the theory.

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